Fundamentals of networks

Inductance of single-layer solenoids*

The approximate value of the *low-frequency* inductance of a single-layer solenoid is†

$L = Fn^2d$ microhenries

where

F = form factor, a function of the ratio d/l. Value of F may be read from the accompanying chart, Fig. 1.

n = number of turns

d = diameter of coil (inches), between centers of conductors

l = length of coil (inches)

= n times the distance between centers of adjacent turns.

The formula is based on the assumption of a uniform current sheet, but the correction due to the use of spaced round wires is usually negligible for practical purposes. For higher frequencies, skin effect alters the inductance slightly. This effect is not readily calculated, but is often negligibly small. However, it must be borne in mind that the formula gives approximately the true value of inductance. In contrast, the *apparent* value is affected by the shunting effect of the distributed capacitance of the coil.

Example: Required a coil of 100 microhenries inductance, wound on a form 2 inches diameter by 2 inches winding length. Then $d/l = 1.00$, and $F = 0.0173$ in Fig. 1.

$$n = \sqrt{\frac{L}{Fd}} = \sqrt{\frac{100}{0.0173 \times 2}} = 54 \text{ turns}$$

Reference to magnet-wire data, Fig. 2, will assist in choosing a desirable size of wire, allowing for a suitable spacing between turns according to the application of the coil. A slight correction may then be made for the increased diameter (diameter of form plus two times radius of wire), if this small correction seems justified.

Approximate formula

For single-layer solenoids of the proportions normally used in radio work, the inductance is given to an accuracy of about 1 percent by

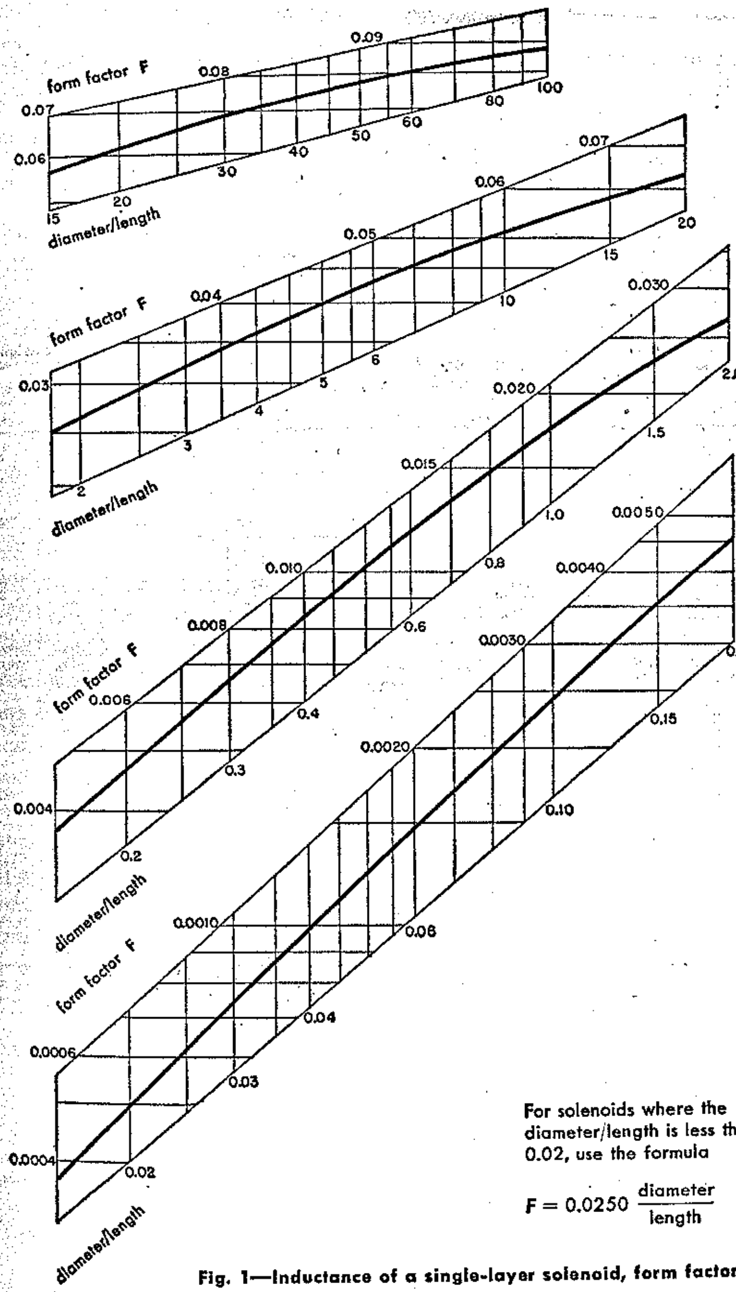
$$L = n^2 \frac{r^2}{9r + 10l} \text{ microhenries}$$

where $r = d/2$.

* Calculation of copper losses in single-layer solenoids is treated in F. E. Terman, "Radio Engineers Handbook," 1st edition, McGraw-Hill Book Company, Inc., New York, N. Y.; 1943: pp. 77–80.

† Formulas and chart (Fig. 1) derived from equations and tables in Bureau of Standards Circular No. C74.

Inductance of single-layer solenoids *continued*



For solenoids where the diameter/length is less than 0.02, use the formula

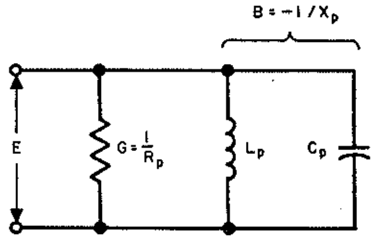
$$F = 0.0250 \frac{\text{diameter}}{\text{length}}$$

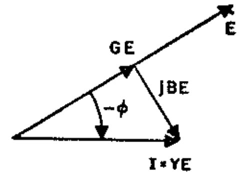
Fig. 1—Inductance of a single-layer solenoid, form factor = F.

Impedance formulas

Parallel and series circuits and their equivalent relationships

Parallel circuit





parallel circuit

Conductance $G = \dfrac{1}{R_p}$

Susceptance $B = -\dfrac{1}{X_p}$

$$= \omega C_p - \frac{1}{\omega L_p}$$

$$\omega = 2\pi f$$

Reactance $X_p = \dfrac{\omega L_p}{1 - \omega^2 L_p C_p}$

Admittance $Y = \dfrac{I}{E} = \dfrac{1}{Z} = G + jB$

$$= \sqrt{G^2 + B^2}\ \angle -\phi = |Y|\ \angle -\phi$$

Impedance $Z = \dfrac{E}{I} = \dfrac{1}{Y}$

$$= \frac{R_p X_p}{R_p^2 + X_p^2}(X_p + jR_p)$$

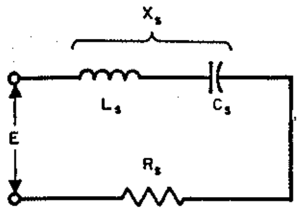
$$= \frac{R_p X_p}{\sqrt{R_p^2 + X_p^2}}\ \angle \phi = |Z|\ \angle \phi$$

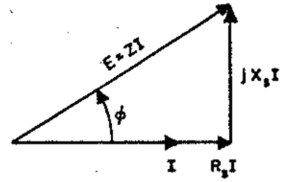
Phase angle $-\phi = \tan^{-1} \dfrac{B}{G}$

$$= \cos^{-1} \frac{G}{|Y|} = -\tan^{-1} \frac{R_p}{X_p}$$

Impedance formulas *continued*

Series circuit





equivalent series circuit

Resistance $= R_s$

Reactance $X_s = \omega L_s - \dfrac{1}{\omega C_s}$

Impedance $Z = \dfrac{E}{I} = R_s + jX_s = \sqrt{R_s^2 + X_s^2}\ \angle \phi = |Z|\ \angle \phi$

Phase angle $\phi = \tan^{-1} \dfrac{X_s}{R_s} = \cos^{-1} \dfrac{R_s}{|Z|}$

For both circuits

Vectors E and I, phase angle ϕ, and Z, Y are identical for the parallel circuit and its equivalent series circuit

$$Q = |\tan \phi| = \frac{|X_s|}{R_s} = \frac{R_p}{|X_p|} = \frac{|B|}{G}$$

$$(\text{pf}) = \cos \phi = \frac{R_s}{|Z|} = \frac{|Z|}{R_p} = \frac{G}{|Y|} = \sqrt{\frac{R_s}{R_p}} = \frac{1}{\sqrt{Q^2 + 1}} = \frac{(\text{kw})}{(\text{kva})}$$

$$Z^2 = R_s^2 + X_s^2 = \frac{R_p^2 X_p^2}{R_p^2 + X_p^2} = R_s R_p = X_s X_p$$

$$Y^2 = G^2 + B^2 = \frac{1}{R_p^2} + \frac{1}{X_p^2} = \frac{G}{R_s}$$

$$R_s = \frac{Z^2}{R_p} = \frac{G}{Y^2} = R_p \frac{X_p^2}{R_p^2 + X_p^2} = R_p \frac{1}{Q^2 + 1}$$

$$X_s = \frac{Z^2}{X_p} = -\frac{B}{Y^2} = X_p \frac{R_p^2}{R_p^2 + X_p^2} = X_p \frac{1}{1 + 1/Q^2}$$

Impedance formulas *continued*

$$R_p = \frac{1}{G} = \frac{Z^2}{R_s} = \frac{R_s^2 + X_s^2}{R_s} = R_s\,(Q^2 + 1)$$

$$X_p = -\frac{1}{B} = \frac{Z^2}{X_s} = \frac{R_s^2 + X_s^2}{X_s} = X_s\left(1 + \frac{1}{Q^2}\right) = \frac{R_sR_p}{X_s} = \pm R_p\sqrt{\frac{R_s}{R_p - R_s}}$$

Approximate formulas

Reactor $R_s = \dfrac{X^2}{R_p}$ and $X = X_s = X_p$ (See Note 1, p. 123)

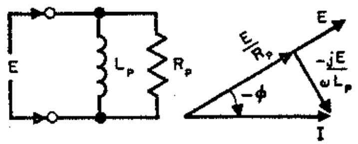
Resistor $R = R_s = R_p$ and $X_s = \dfrac{R^2}{X_p}$ (See Note 2, p. 123)

Simplified parallel and series circuits

$$X_p = \omega L_p \qquad B = -\frac{1}{\omega L_p} \qquad X_s = \omega L_s$$

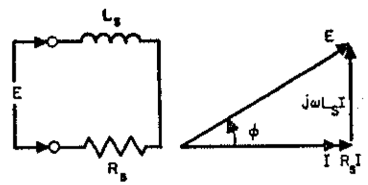
$$\tan\phi = \frac{\omega L_s}{R_s} = \frac{R_p}{\omega L_p}$$

$$Q = \frac{\omega L_s}{R_s} = \frac{R_p}{\omega L_p}$$



$$(\text{pf}) = \frac{R_s}{\sqrt{R_s^2 + \omega^2 L_s^2}}$$

$$= \frac{\omega L_p}{\sqrt{R_p^2 + \omega^2 L_p^2}}$$



$(\text{pf}) = \dfrac{1}{Q}$ approx (See Note 3, p. 123)

$$R_s = R_p\frac{1}{Q^2 + 1} \qquad R_p = R_s\,(Q^2 + 1) \qquad Z = R_p\frac{1 + jQ}{1 + Q^2}$$

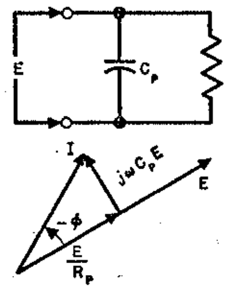
$$L_s = L_p\frac{1}{1 + 1/Q^2} \qquad L_p = L_s\left(1 + \frac{1}{Q^2}\right) \qquad Y = \frac{1}{R_s}\frac{1 - jQ}{1 + Q^2}$$

Impedance formulas *continued*

$$X_p = \frac{-1}{\omega C_p} \qquad B = \omega C_p \qquad X_s = \frac{-1}{\omega C_s}$$

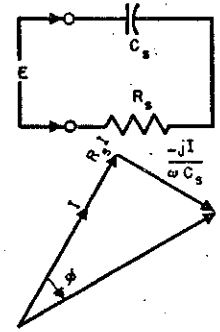
$$\tan\phi = \frac{-1}{\omega C_s R_s} = -\omega C_p R_p$$

$$Q = \frac{1}{\omega C_s R_s} = \omega C_p R_p$$



$$(\text{pf}) = \frac{\omega C_s R_s}{\sqrt{1 + \omega^2 C_s^2 R_s^2}} = \frac{1}{\sqrt{1 + \omega^2 C_p^2 R_p^2}}$$

$(\text{pf}) \approx \dfrac{1}{Q}$ (See Note 3)



$$R_s = R_p\frac{1}{Q^2 + 1} \qquad R_p = R_s\,(Q^2 + 1)$$

$$C_s = C_p\left(1 + \frac{1}{Q^2}\right) \qquad C_p = C_s\frac{1}{1 + 1/Q^2}$$

$$Z = R_p\frac{1 - jQ}{1 + Q^2} \qquad Y = \frac{1}{R_s}\frac{1 + jQ}{1 + Q^2}$$

Approximate formulas

Inductor $R_s = \omega^2L^2/R_p$ and $L = L_p = L_s$ (See Note 1)

Resistor $R = R_s = R_p$ and $L_p = R^2/\omega^2 L_s$ (See Note 2)

Capacitor $R_s = 1/\omega^2C^2R_p$ and $C = C_p = C_s$ (See Note 1)

Resistor $R = R_s = R_p$ and $C_s = 1/\omega^2 C_p R^2$ (See Note 2)

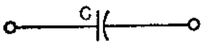
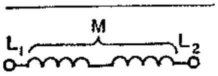
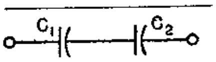
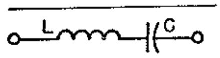
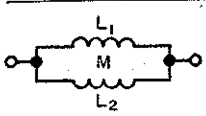
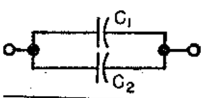
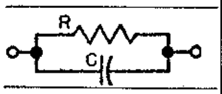
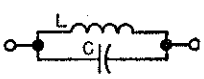
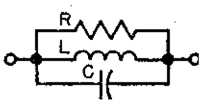
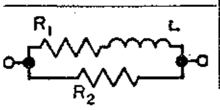
Note 1: (Small resistive component) Error in percent $= -100/Q^2$ (for $Q = 10$, error = 1 percent low)

Note 2: (Small reactive component) Error in percent $= -100\,Q^2$ (for $Q = 0.1$, error = 1 percent low)

Note 3: Error in percent $= +50/Q^2$ approximately (for $Q = 7$, error = 1 percent high)

Impedance $Z = R + jX$ **ohms** — **phase angle** $\phi = \tan^{-1}\frac{X}{R}$

magnitude $|Z| = [R^2 + X^2]^{\frac{1}{2}}$ **ohms** — **admittance** $Y = \frac{1}{Z}$ **mhos**

diagram	impedance Z	magnitude |Z|	phase angle ϕ	admittance Y
	R	R	0	$\frac{1}{R}$
	$j\omega L$	ωL	$+\frac{\pi}{2}$	$-j\frac{1}{\omega L}$
	$-j\frac{1}{\omega C}$	$\frac{1}{\omega C}$	$-\frac{\pi}{2}$	$j\omega C$
	$j\omega\ (L_1 + L_2 \pm 2M)$	$\omega(L_1 + L_2 \pm 2M)$	$+\frac{\pi}{2}$	$-j\frac{1}{\omega\ (L_1 + L_2 \pm 2M)}$
	$-j\frac{1}{\omega}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)$	$\frac{1}{\omega}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)$	$-\frac{\pi}{2}$	$j\omega\frac{C_1C_2}{C_1+C_2}$
	$R + j\omega L$	$[R^2+\omega^2L^2]^{\frac{1}{2}}$	$\tan^{-1}\frac{\omega L}{R}$	$\frac{R - j\omega L}{R^2+\omega^2L^2}$
	$R - j\frac{1}{\omega C}$	$\frac{1}{\omega C}[1+\omega^2C^2R^2]^{\frac{1}{2}}$	$-\tan^{-1}\frac{1}{\omega CR}$	$\frac{R + j\frac{1}{\omega C}}{R^2+\frac{1}{\omega^2C^2}}$
	$j\left(\omega L - \frac{1}{\omega C}\right)$	$\left(\omega L - \frac{1}{\omega C}\right)$	$\pm\frac{\pi}{2}$	$j\frac{\omega C}{1-\omega^2LC}$
	$R + j\left(\omega L - \frac{1}{\omega C}\right)$	$\left[R^2+\left(\omega L - \frac{1}{\omega C}\right)^2\right]^{\frac{1}{2}}$	$\tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$	$\frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}$
	$\frac{R_1R_2}{R_1+R_2}$	$\frac{R_1R_2}{R_1+R_2}$	0	$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
	$j\omega\left[\frac{L_1L_2-M^2}{L_1+L_2\mp 2M}\right]$	$\omega\left[\frac{L_1L_2-M^2}{L_1+L_2\mp 2M}\right]$	$+\frac{\pi}{2}$	$-j\frac{1}{\omega}\left[\frac{L_1+L_2\mp 2M}{L_1L_2-M^2}\right]$
	$-j\frac{1}{\omega\ (C_1+C_2)}$	$\frac{1}{\omega\ (C_1+C_2)}$	$-\frac{\pi}{2}$	$j\omega(C_1+C_2)$
	$\omega LR\left[\frac{\omega L + jR}{R^2+\omega^2L^2}\right]$	$\frac{\omega LR}{[R^2+\omega^2L^2]^{\frac{1}{2}}}$	$\tan^{-1}\frac{R}{\omega L}$	$\frac{1}{R} - j\frac{1}{\omega L}$
	$\frac{R(1 - j\omega CR)}{1+\omega^2C^2R^2}$	$\frac{R}{[1+\omega^2C^2R^2]^{\frac{1}{2}}}$	$-\tan^{-1}\omega CR$	$\frac{1}{R} + j\omega C$
	$j\frac{\omega L}{1-\omega^2LC}$	$\frac{\omega L}{1-\omega^2LC}$	$\pm\frac{\pi}{2}$	$j\left(\omega C - \frac{1}{\omega L}\right)$
	$\frac{\frac{1}{R} - j\left(\omega C - \frac{1}{\omega L}\right)}{\left(\frac{1}{R}\right)^2+\left(\omega C - \frac{1}{\omega L}\right)^2}$	$\frac{1}{\left[\left(\frac{1}{R}\right)^2+\left(\omega C - \frac{1}{\omega L}\right)^2\right]^{\frac{1}{2}}}$	$\tan^{-1} R\left(\frac{1}{\omega L} - \omega C\right)$	$\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$
	$R_2\frac{R_1(R_1+R_2)+\omega^2L^2+j\omega LR_2}{(R_1+R_2)^2+\omega^2L^2}$	$R_2\left[\frac{R_1^2+\omega^2L^2}{(R_1+R_2)^2+\omega^2L^2}\right]^{\frac{1}{2}}$	$\tan^{-1}\frac{\omega LR_2}{R_1\ (R_1+R_2)+\omega^2L^2}$	$\frac{R_1(R_1+R_2)+\omega^2L^2-j\omega LR_2}{R_2(R_1^2+\omega^2L^2)}$

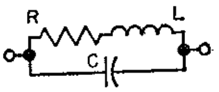
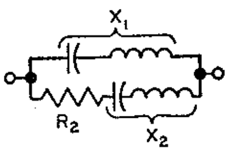
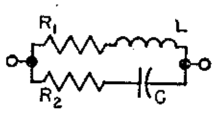
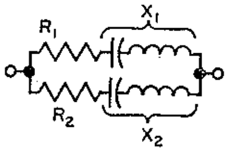
continued **Impedance formulas**

impedance $Z = R + jX$ ohms

phase angle $\phi = \tan^{-1}\frac{X}{R}$

magnitude $|Z| = [R^2 + X^2]^{\frac{1}{2}}$ ohms

admittance $Y = \frac{1}{Z}$ mhos

	impedance Z	$\frac{R + j\omega[L(1-\omega^2LC) - CR^2]}{(1-\omega^2LC)^2 + \omega^2C^2R^2}$
	magnitude $\|Z\|$	$\left[\frac{R^2 + \omega^2L^2}{(1-\omega^2LC)^2 + \omega^2C^2R^2}\right]^{\frac{1}{2}}$
	phase angle ϕ	$\tan^{-1}\frac{\omega[L(1-\omega^2LC) - CR^2]}{R}$
	admittance Y	$\frac{R - j\omega[L(1-\omega^2LC) - CR^2]}{R^2 + \omega^2L^2}$
	impedance Z	$X_1\frac{X_1R_2 + j[R_2^2 + X_2(X_1+X_2)]}{R_2^2 + (X_1+X_2)^2}$
	magnitude $\|Z\|$	$X_1\left[\frac{R_2^2 + X_2^2}{R_2^2 + (X_1+X_2)^2}\right]^{\frac{1}{2}}$
	phase angle ϕ	$\tan^{-1}\frac{R_2^2 + X_2(X_1+X_2)}{X_1R_2}$
	admittance Y	$\frac{R_2X_1 - j(R_2^2 + X_2^2 + X_1X_2)}{X_1(R_2^2 + X_2^2)}$
	impedance Z	$\frac{R_1R_2(R_1+R_2) + \omega^2L^2R_2 + \frac{R_1}{\omega^2C^2}}{(R_1+R_2)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} + j\frac{\omega LR_2^2 - \frac{R_1^2}{\omega C} - \frac{L}{C}\left(\omega L - \frac{1}{\omega C}\right)}{(R_1+R_2)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ Note: When $R_1 = R_2 = \sqrt{L/C}$, then $Z = R_1 = R_2$, a pure resistance at any frequency where the given conditions hold. Compare Case 3a, p. 156.
	magnitude $\|Z\|$	$\left[\frac{(R_1^2 + \omega^2L^2)\left(R_2^2 + \frac{1}{\omega^2C^2}\right)}{(R_1+R_2)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\right]^{\frac{1}{2}}$
	phase angle ϕ	$\tan^{-1}\left[\frac{\omega LR_2^2 - \frac{R_1^2}{\omega C} - \frac{L}{C}\left(\omega L - \frac{1}{\omega C}\right)}{R_1R_2(R_1+R_2) + \omega^2L^2R_2 + \frac{R_1}{\omega^2C^2}}\right]$
	admittance Y	$\frac{R_1 + \omega^2C^2R_1R_2(R_1+R_2) + \omega^4L^2C^2R_2}{(R_1^2+\omega^2L^2)(1+\omega^2C^2R_2^2)} + j\omega\left[\frac{CR_1^2 - L + \omega^2LC(L - CR_2^2)}{(R_1^2+\omega^2L^2)(1+\omega^2C^2R_2^2)}\right]$
	impedance Z	$\frac{(R_1R_2 - X_1X_2) + j(R_1X_2 + R_2X_1)}{(R_1+R_2) + j(X_1+X_2)}$
	magnitude $\|Z\|$	$\left[\frac{(R_1^2+X_1^2)(R_2^2+X_2^2)}{(R_1+R_2)^2 + (X_1+X_2)^2}\right]^{\frac{1}{2}}$
	phase angle ϕ	$\tan^{-1}\frac{X_1}{R_1} + \tan^{-1}\frac{X_2}{R_2} - \tan^{-1}\frac{X_1+X_2}{R_1+R_2}$
	admittance Y	$\frac{1}{R_1 + jX_1} + \frac{1}{R_2 + jX_2}$

Skin effect

Symbols

A = correction coefficient

D = diameter of conductor in inches

f = frequency in cycles/second

R_{ac} = resistance at frequency f

R_{dc} = direct-current resistance

R_{sq} = resistance per square

T = thickness of tubular conductor in inches

T_1 = depth of penetration of current

δ = skin depth

λ = free-space wavelength in meters

μ_r = relative permeability of conductor material ($\mu_r = 1$ for copper and other nonmagnetic materials)

ρ = resistivity of conductor material at any temperature

ρ_c = resistivity of copper at 20 degrees centigrade = 1.724 microhm-centimeter

Skin depth

The skin depth is that distance below the surface of a conductor where the current density has diminished to $1/e$ of its value at the surface. The thickness of the conductor is assumed to be several (perhaps at least three) times the skin depth. Imagine the conductor replaced by a cylindrical shell of the same surface shape but of thickness equal to the skin depth; with uniform current density equal to that which exists at the surface of the actual conductor. Then the total current in the shell and its resistance are equal to the corresponding values in the actual conductor.

The skin depth and the resistance per square (of any size), in meter–kilogram–second (rationalized) units, are

$$\delta = (\lambda/\pi\sigma\mu c)^{1/2} \text{ meter}$$

$$R_{sq} = 1/\delta\sigma \text{ ohm}$$

where

c = velocity of light in vacuo = 2.998×10^8 meters/second

$\mu = 4\pi \times 10^{-7}\ \mu_r$ henry/meter

$1/\sigma = 1.724 \times 10^{-8}\ \rho/\rho_c$ ohm-meter

Skin effect *continued*

For numerical computations:

$$\delta = (3.82 \times 10^{-4}\ \lambda^{1/2})\ k_1 = (6.61/f^{1/2})k_1 \text{ centimeter}$$

$$\delta = (1.50 \times 10^{-4}\ \lambda^{1/2})\ k_1 = (2.60/f^{1/2})k_1 \text{ inch}$$

$$\delta_m = (2.60/f_{mc}^{1/2})\ k_1 \text{ mils}$$

$$R_{sq} = (4.52 \times 10^{-3}/\lambda^{1/2})\ k_2 = (2.61 \times 10^{-7}\ f^{1/2})\ k_2 \text{ ohm}$$

where

$$k_1 = [(1/\mu_r)\ \rho/\rho_c]^{1/2}$$

$$k_2 = (\mu_r\rho/\rho_c)^{1/2}$$

k_1, k_2 = unity for copper

Example: What is the resistance/foot of a cylindrical copper conductor of diameter D inches?

$$R = \frac{12}{\pi D} R_{sq} = \frac{12}{\pi D} \times 2.61 \times 10^{-7}\ (f)^{1/2}$$

$$= 0.996 \times 10^{-6}\ (f)^{1/2}/D \text{ ohm/foot}$$

If

$D = 1.00$ inch

$f = 100 \times 10^6$ cycles/second,

$R = 0.996 \times 10^{-6} \times 10^4 \approx 1 \times 10^{-2}$ ohm/foot.

General considerations

Fig. 7 shows the relationship of R_{ac}/R_{dc} versus $D\sqrt{f}$ for copper, or versus $D\sqrt{f}\ \sqrt{\mu_r\rho_c/\rho}$ for any conductor material, for an isolated straight solid conductor of circular cross section. Negligible error in the formulas for R_{ac} results when the conductor is spaced at least $10D$ from adjacent conductors. When the spacing between axes of parallel conductors carrying the same current is $4D$, the resistance R_{ac} is increased about 3 percent, when the depth of penetration is small. The formulas are accurate for concentric lines due to their circular symmetry.

For values of $D\sqrt{f}\ \sqrt{\mu_r\rho_c/\rho}$ greater than 40,

$$\frac{R_{ac}}{R_{dc}} = 0.0960\ D\sqrt{f}\ \sqrt{\mu_r\rho_c/\rho} + 0.26 \quad (1)$$